# Chapter 10
# An Introduction to Wavelet Analysis

[*This chapter is based on the lectures of Professor D.V. Pai, Department of Mathematics, Indian Institute of Technology Bombay, Powai, Mumbai - 400 076, India.*]

## 10.0 Introduction

During the last 20 years or so, the subject of "Wavelet analysis" has attracted a lot of attention from both mathematicians and engineers alike. Vaguely speaking the term "Wavelet" means a little wave, and it includes functions that are reasonably localized in the *time domain* as well as in the *frequency domain*. The idea seems to evolve from the limitation imposed by the *uncertainty principle* of Physics which puts a limit on simultaneous localization in both the time and the frequency domains.

From a historical perspective, although the idea of wavelet seems to originate with the work by Gabor and by Neumann in the late 1940s, this term seems to have been coined for the first time in the more recent seminal paper of Grossman and Morlet (1984). Nonethless, the techniques based on the use of translations and dilations are much older. This can be at least traced back to Calderón (1964) in his study of singular integral operators. Starting with the pioneering works reported in the early monographs contributed by Meyer (1992), Mallat (1989), Chui (1992), Daubechies (1992) and others, an ever increasing number of books, monographs and proceedings of international conferences which have appeared more recently in this field only point to its growing importance.

The main aim of these lectures is to attempt to present a quick introduction of this field to a beginner. We will mainly emphasize here the construction of orthonormal (o.n.) wavelets using the so-called *two-scale relation* . This will lead us to a natural classification of wavelets as well as to the classical multiresolution analysis. In particular, we will also attempt to highlight the spline wavelets of Chui and Wang (1993).

## 10.1 Fourier Analysis to Wavelet Analysis

Let $L^2(0,2\pi)$ = the space of all (equivalence classes) of $2\pi$-periodic, Lebesgue measurable functions $f : \mathbb{R} \to \mathbb{C}$ such that $\int_0^{2\pi} |f(t)|^2 \mathrm{d}t < \infty$. $L^2(0,2\pi)$ is a Hilbert space furnished with the inner product

$$(f,g) = \frac{1}{2\pi}\int_0^{2\pi} f(t)\overline{g(t)}\mathrm{d}t, \quad f,g \in L^2(0,2\pi)$$

and the corresponding norm

$$\|f\|_2 = \left\{\frac{1}{2\pi}\int_0^{2\pi} |f(t)|^2 \mathrm{d}t\right\}^{1/2}.$$

Any $f$ in $L^2(0,2\pi)$ has a *Fourier series representation*

$$f(t) = \sum_{k=-\infty}^{\infty} c_k \mathrm{e}^{ikt}, \tag{10.1.1}$$

where the constants $c_k$, called the *Fourier coefficients* of $f$, are defined by

$$c_k = (f,w_k) = \frac{1}{2\pi}\int_0^{2\pi} f(t)\bar{\mathrm{e}}^{ikt}\mathrm{d}t, \quad w_k(t) = \mathrm{e}^{ikt}. \tag{10.1.2}$$

This is a consequence of the important fact that $\{w_k(t) : k \in \mathbb{Z}\}$ is an *orthonormal basis* of $L^2(0,2\pi)$. Also recall that the Fourier series representation satisfies the so-called *Parseval identity*:

$$\|f\|_2^2 = \sum_{k=-\infty}^{\infty} |c_k|^2, \quad f \in L^2(0,2\pi).$$

Let us emphasize two interesting features in the Fourier series representation (10.1.1). Firstly, note that $f$ is decomposed into an infinite sum of mutually orthogonal components $c_k w_k$. The second interesting feature to be noted of (10.1.1) is that the o.n. basis $\{w_k : k \in \mathbb{Z}\}$ is generated by "dilates" of a single function

$$w(t) := w_1(t) = \mathrm{e}^{it};$$

that is, $w_k(t) = w(kt),\ k \in \mathbb{Z}$, is , in fact, an *integral dilate* of $w(t)$. Let us reemphasize the following remarkable fact:

*Every $2\pi$-periodic square-integrable function is generated by a superposition of integral dilates of the single basic function $w(t) = \mathrm{e}^{it}$.*

The basic function $w(t) = \cos t + i \sin t$ is a sinusoidal wave. For any integer $k$ with $|k|$ large, the wave $w_k(t) = w(kt)$ has high *frequency*, and for $k$ in $\mathbb{Z}$ with $|k|$ small, the wave $w_k$ has low frequency. Thus *every function in $L^2(0,2\pi)$ is composed of waves with various frequencies.*

Let $L^2(\mathbb{R})$ := the space of all (equivalence classes) of complex measurable functions, defined on $\mathbb{R}$ for which

$$\int_{\mathbb{R}} |f(t)|^2 \mathrm{d}t < \infty.$$

Note that the space $L^2(\mathbb{R})$ is a Hilbert space with the inner product

$$\langle f, g\rangle = \int_{\mathbb{R}} f(t)\overline{g(t)}\mathrm{d}t \quad f, g \in L^2(\mathbb{R})$$

and the norm

$$\|f\|_2 = \left\{\int_{\mathbb{R}} |f(t)|^2 \mathrm{d}t\right\}^{1/2} \quad f \in L^2(\mathbb{R}).$$

*Wavelet analysis* also begins with a quest for a single function $\psi$ in $L^2(\mathbb{R})$ to generate $L^2(\mathbb{R})$. Since any such function must decay to zero at $\pm\infty$, we must give up, as being too restrictive, the idea of using only linear combinations of dilates of $\psi$ to recover $L^2(\mathbb{R})$. Instead, it is natural to consider both the *dilates* and the *translates*. The most convenient family of functions for this purpose is thus given by

$$\psi_{j,k}(t) = 2^{j/2}\psi(2^j t - k), \quad j, k \in \mathbb{Z}. \tag{10.1.3}$$

This involves a *binary dilation* (dilation by $2^j$) and a *dyadic translation* (of $k/2^j$).

**Lemma 10.1.1:** Let $\phi, \psi$ be in $L^2(\mathbb{R})$. Then, for $i, j, k, \ell$ in $\mathbb{Z}$, we have:

(i) $\langle \psi_{j,k}, \phi_{j,\ell}\rangle = \langle \psi_{i,k}, \phi_{i,\ell}\rangle$;
(ii) $\|\psi_{j,k}\|_2 = \|\psi\|_2$.

**Proof 10.1.1:**
(i) We have

$$\langle \psi_{j,k}, \phi_{j,\ell}\rangle = \int_{\mathbb{R}} 2^{j/2}\psi(2^j t - k)2^{j/2}\overline{\phi(2^j t - \ell)}\mathrm{d}t.$$

Put $t = 2^{i-j}x$ to get

$$\begin{aligned} \text{R.H.S.} &= \int_{\mathbb{R}} 2^{i/2}\psi(2^i x - k)2^{i/2}\overline{\phi(2^i x - \ell)}\mathrm{d}x \\ &= \langle \psi_{i,k}, \phi_{i,\ell}\rangle. \end{aligned}$$

Note that

$$\begin{aligned} \|\psi_{j,k}\|_2^2 &= 2^j \int_{\mathbb{R}} |\psi(2^j t - k)|^2 \mathrm{d}t \\ &= \int_{\mathbb{R}} |\psi(x)|^2 \mathrm{d}x = \|\psi\|_2^2. \text{ (We put } x = 2^j t - k.\text{)} \end{aligned}$$

**Remark 10.1.1:** For $i, j \in \mathbb{Z}$, we have:
the set $\{\psi_{i,k} : k \in \mathbb{Z}\}$ is orthonormal
$\Leftrightarrow$ the set $\{\psi_{j,k} : k \in \mathbb{Z}\}$ is orthonormal.

**Definition 10.1.1.** A function $\psi \in L^2(\mathbb{R})$ is called an **orthonormal wavelet** (or an **o.n. wavelet)** if the family $\{\psi_{j,k}\}$, as defined in (10.1.3), is an orthonormal basis of $L^2(\mathbb{R})$; that is,

$$\langle \psi_{j,k}, \psi_{i,\ell} \rangle = \delta_{j,i}\delta_{k,\ell}, \quad j,k,i,\ell \in \mathbb{Z} \tag{10.1.4}$$

and every $f$ in $L^2(\mathbb{R})$ has a representation

$$f(t) = \sum_{j,k=-\infty}^{\infty} c_{j,k}\psi_{j,k}(t), \tag{10.1.5}$$

where the convergence of the series in (10.1.5) is in $L^2(\mathbb{R})$:

$$\lim_{\substack{M_1,N_1,M_2,N_2 \\ \to\infty}} \left\| f - \sum_{j=-M_1}^{N_1} \sum_{k=-M_2}^{N_2} c_{j,k}\psi_{j,k} \right\|_2 = 0.$$

The series representation (10.1.5) of $f$ is called a **wavelet series** and the coefficients $c_{j,k}$ given by

$$c_{j,k} = \langle f, \psi_{j,k} \rangle \tag{10.1.6}$$

are called the **wavelet coefficients**.

**Example 10.1.1.** Let us recall the definition of the **Haar function** $\psi^H(t)$ given below:

$$\psi^H(t) := \begin{cases} 1, & \text{if } 0 \le t < \frac{1}{2} \\ -1, & \text{if } \frac{1}{2} \le t < 1 \\ 0, & \text{otherwise.} \end{cases}$$

At this stage, the reader is urged as an exercise to verify that the family $\{\psi^H_{j,k} : j,k \in \mathbb{Z}\}$ is orthonormal in the space $L^2(\mathbb{R})$. We will come back again to this example in the next section. It will be shown there that $\psi^H$ is, in fact, an o.n. wavelet.

Next, let us recall that the **Fourier transform** of a function $f$ in $L^1(\mathbb{R})$ is the function $\hat{f}$ defined by

$$\hat{f}(w) := \int_{\mathbb{R}} f(t)\bar{e}^{iwt}\mathrm{d}t, \quad w \in \mathbb{R}.$$

**Definition 10.1.2.** If a function $\psi \in L^2(\mathbb{R})$ satisfies the **admissibility condition:**

$$C_\psi := \int_{\mathbb{R}} \frac{|\hat{\psi}(w)|^2}{|w|}\mathrm{d}w < \infty$$

then $\psi$ is called a **"basic wavelet"**.

The definition is due to Grossman and Morlet (1984). It is related to the invertibility of the continuous wavelet transform as given by the next definition.

**Definition 10.1.3.** Relative to every basic wavelet $\psi$, consider the family of wavelets defined by

$$\psi_{a,b}(t) := |a|^{-1/2}\psi\left(\frac{t-b}{a}\right), a,b \in \mathbb{R}, \quad a \neq 0. \tag{10.1.7}$$

The **continuous wavelet transform** (CWT) corresponding to $\psi$ is defined by

$$\begin{aligned}(W_\psi f)(a,b) &= |a|^{-1/2}\int_{\mathbb{R}} f(t)\overline{\psi\left(\frac{t-b}{a}\right)}\mathrm{d}t, \quad f \in L^2(\mathbb{R}) \\ &= \langle f, \psi_{a,b}\rangle.\end{aligned} \tag{10.1.8}$$

Let us note that the wavelet coefficients in (10.1.7) and (10.1.8) become

$$c_{j,k} = (W_\psi f)\left(\frac{1}{2^j}, \frac{k}{2^j}\right). \tag{10.1.9}$$

Thus wavelet series and the continuous wavelet transform are intimately related.

Let us also state the following *inversion theorem* for the continuous wavelet transform. The proof uses the Fourier transform of $\psi_{a,b}$, the Parseval identity and the fact that the Gaussian functions

$$g_\alpha(t) := \frac{1}{2\sqrt{\pi\alpha}}\mathrm{e}^{-\frac{t^2}{4\alpha}}, \;\; \alpha > 0$$

is an approximate identity in $L^1(\mathbb{R})$. Thus, for $f \in L^1(\mathbb{R})$, $\lim_{\alpha\to 0}(f \cdot g_\alpha)(t) = f(t)$ at every point $t$ where $f$ is continuous. The details of the proof are left to the reader as an exercise.

**Theorem 10.1.1.** *Let $\psi$ in $L^2(\mathbb{R})$ be a basic wavelet which defines a continuous wavelet transform $W_\psi$. Then for any $f$ in $L^2(\mathbb{R})$ and $t \in \mathbb{R}$ at which $f$ is continuous,*

$$f(t) = \frac{1}{C_\psi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(W_\psi f)(a,b)\psi_{a,b}(t)\frac{\mathrm{d}a\mathrm{d}b}{a^2}, \tag{10.1.10}$$

*where $\psi_{a,b}$ is defined by (10.1.7).*

## 10.2 Construction of Orthonormal Wavelets

One of the first examples of an o.n. wavelet is due to Haar (1910). Let us recall (Example 10.1.1) that it is called the *Haar function* defined by

$$\psi^H(t) := \begin{cases} 1, \text{ if } & 0 \leq t < \frac{1}{2} \\ -1, \text{ if } & \frac{1}{2} \leq t < 1 \\ 0, \text{ otherwise.} \end{cases}$$

Most of the recent theories on wavelets are no doubt inspired by this example. However, as it turns out, its discontinuous nature is a serious drawback in many applications. Thus, in these lectures, one of our quests is to explore more examples. Let us begin with a real function $\phi$ in $L^2(\mathbb{R})$. As a first step let us assume that

$(S_o)$ : the family $\{\phi_{o,k}(t) = \phi(t-k) : k \in \mathbb{Z}\}$ is orthonormal.

Then it follows from Lemma 10.1.1, that the family $\{\phi_{j,k}(t) : k \in \mathbb{Z}\}$ is orthonormal, for each $j \in \mathbb{Z}$. Let us define

$$V_j = \overline{\text{span}}\{\phi_{j,k} : k \in \mathbb{Z}\}, \quad (j \in \mathbb{Z}),$$

the closure being taken in the topology of $L^2(\mathbb{R})$. It results from the next lemma that

$$V_j = \left\{ \sum_{k \in \mathbb{Z}} c_k \phi_{j,k} : c = \{c_k\} \in \ell^2(\mathbb{Z}) \right\}. \tag{10.2.1}$$

**Lemma 10.2.1:** Let $\{u_k : k \in \mathbb{Z}\}$ be an orthonormal bi-infinite sequence in a Hilbert space $X$. Then

$$\overline{\text{span}}\{u_k : k \in \mathbb{Z}\} = \left\{ \sum_{k=-\infty}^{\infty} c_k u_k : c = \{c_k\} \in \ell^2(\mathbb{Z}) \right\}.$$

**Proof 10.2.1:** Let $V$ denote the L.H.S. set and $U$ be the R.H.S. set. Note that for a sequence $\{c_k\} \in \ell^2(\mathbb{Z})$, the series $\sum_k c_k u_k$ converges, because its partial sums form a Cauchy sequence in $X$:

$$\left\| \sum_{k=-M}^{M} c_k u_k - \sum_{k=-N}^{N} c_k u_k \right\|^2 = \sum_{k=N+1}^{M} |c_k|^2 + \sum_{k=-M}^{-(N+1)} |c_k|^2 \to 0 \text{ as } M, N \to \infty.$$

Clearly, $U \subset V$. Also, $U$ is a closed subspace being isometrically isomorphic to $\ell^2(\mathbb{Z})$. Since $u_k \in U$, we have $\text{span}\{u_k\} \subset U \Rightarrow \overline{\text{span}}\{u_k\} \subset \overline{U} = U$. Hence $V \subset U$

Let us assume, *in addition*, that $\phi \in V_1$. Then for a suitable $c \in \ell^2(\mathbb{Z})$, we will have

$(S_1)$ $\qquad \phi(t) = \sum_{k \in \mathbb{Z}} c_k \phi(2t-k).$

This is called a **two-scale relation** or a **dilation equation**.

**Lemma 10.2.2:** Let $\phi$ in $L^2(\mathbb{R})$ satisfy $(S_0)$ and $(S_1)$. Then for all $i, j$ in $\mathbb{Z}$,

$$\sum_k \left[ c_{j-2k} c_{i-2k} + (-1)^{i+j} c_{1-j+2k} c_{1-i+2k} \right] = 2\delta_{j,i}. \tag{10.2.2}$$

Here the coefficients $c_j$'s are as defined in $(S_1)$.

**Proof 10.2.2:**

**Case 1:** $i+j$ is odd.

Then L.H.S. of (10.2.2) becomes

$$\sum_k c_{j-2k}c_{i-2k} - \sum_k c_{1-j+2k}c_{1-i+2k}.$$

Put $k=-r$ in the first sum and $k=r+\frac{i+j-1}{2}$ in the second sum to obtain

$$\sum_r c_{j+2r}c_{i+2r} - \sum_r c_{i+2r}c_{j+2r} = 0.$$

**Case 2:** $i+j$ is even.

Then L.H.S. of (10.2.2) becomes $\sum_k c_{j-2k}c_{i-2k} + \sum_k c_{1-j+2k}c_{1-i+2k}$. Put $k=-r$ in the first sum and $k=r+\frac{i+j}{2}$ in the second sum to obtain

$$\sum_r c_{j+2r}c_{i+2r} + \sum_r c_{i+2r+1}c_{j+2r+1} = \sum_k c_{j+k}c_{i+k} = \sum_\ell c_\ell c_{\ell+i-j}.$$

(We put $j+k=\ell$.)

Since the set $\{\phi_{1,k} : k \in \mathbb{Z}\}$ is orthonormal,

$$\sum_\ell c_\ell c_{\ell+i-j} = \langle \sum_\ell c_\ell \phi_{1,\ell}, \sum_\ell c_{\ell+i-j}\phi_{1\ell}\rangle.$$

Using $(S_1)$, we have

$$\sum_\ell c_\ell \phi_{1\ell}(t) = \sum_\ell c_\ell 2^{1/2}\phi(2t-\ell) = 2^{1/2}\phi(t).$$

Likewise,

$$\begin{aligned}\sum_\ell c_{\ell+i-j}\phi_{1\ell}(t) &= \sum_\ell c_{\ell+i-j}2^{1/2}\phi(2t-\ell)(\text{put } \ell+i-j=m)\\ &= \sum_m c_m 2^{1/2}\phi(2t+i-j-m)\\ &= 2^{1/2}\sum_m c_m\phi(2(t-\frac{j-i}{2})-m)\\ &= 2^{1/2}\phi(t-\frac{j-i}{2}) = 2^{1/2}\phi_{0,\frac{j-i}{2}}.\end{aligned}$$

Thus,

$$\begin{aligned}\sum_\ell c_\ell c_{\ell+i-j} &= \langle 2^{1/2}\phi_{0,0}, 2^{1/2}\phi_{0,\frac{j-i}{2}}\rangle\\ &= 2\delta_{j,i}, \text{ using } (S_0).\end{aligned}$$

**Lemma 10.2.3:** Let $\phi$ in $L^2(\mathbb{R})$ satisfy $(S_0)$ and $(S_1)$. Let $\psi$ in $L^2(\mathbb{R})$ be defined by

$$(W) \qquad \psi(t) = \sum_{k=-\infty}^{\infty} (-1)^k l c_{1-k} \phi(2t-k).$$

Then for $k \in \mathbb{Z}$,

$$\phi_{1,k} = 2^{-1/2} \sum_m \left[ c_{k-2m}\phi_{0,m} + (-1)^k c_{1-k+2m} \psi_{0,m} \right]. \tag{10.2.3}$$

Here the coefficients $c_j$'s are as defined in the two-scale relation $(S_1)$.

**Proof 10.2.3:** By $(S_1)$, $\qquad \phi(t) = \sum_i c_i \phi(2t-i)$

$$\Rightarrow \quad \phi(t-m) = \sum_i c_i \phi(2(t-m)-i) = \sum_i 2^{-1/2} c_i 2^{1/2} \phi(2t-2m-i)$$

$$\Rightarrow \quad \phi_{0,m} = \sum_i 2^{-1/2} c_i \phi_{1,2m+i}.$$

Likewise,

$$\psi_{0,m} = \sum_i (-1)^i 2^{-1/2} c_{1-i} \phi_{1,2m+i}.$$

The (R.H.S.) of (10.2.3) can now be written as

$$2^{-1/2} \sum_m \left[ C_{k-2m} \sum_i 2^{-1/2} c_i \phi_{1,2m+i} + (-1)^k c_{1-k+2m} \sum_i 2^{-1/2} (-1)^i c_{1-i} \phi_{1,2m+i} \right].$$

Changing the index $i$ to $r$ by the equation $r = 2m+i$, one obtains:

$$2^{-1} \sum_r \sum_m \left[ c_{k-2m} c_{r-2m} + (-1)^{k+r} c_{1-k+2m} c_{1-r+2m} \right] \phi_{1,r} = 2^{-1} \sum_r 2\delta_{k,r} \phi_{1,r}.$$

The last expression equals $\phi_{1,k}$.

**Lemma 10.2.4:** Let $U$ and $V$ be closed subspaces of a Hilbert space $X$ such that $U \perp V$. Then $U+V$ is closed.

**Proof 10.2.4:** Let $w_n = u_n + v_n, u_n \in U, v_n \in V$, be such that $w_n \to w$. Since

$$\|w_n - w_m\|^2 = \|u_n - u_m\|^2 + \|v_n - v_m\|^2$$

and $\{w_n\}$ is Cauchy, both the sequences $\{u_n\}, \{u_n\}$ are Cauchy. If $u = \lim u_n, v = \lim v_n$, then

$$w = u + v \in U + V.$$

**Theorem 10.2.1.** *Let $\phi$ in $L^2(\mathbb{R})$ satisfy properties $(S_0)$ and $(S_1)$. Let $\psi$ in $L^2(\mathbb{R})$ be defined by $(W)$. Then the family $\{\psi_{j,k} : j,k \in \mathbb{Z}\}$ is orthonormal. Moreover, $V_1 = V_0 \bigoplus^{\perp} W_0$, where*

$$W_0 = \overline{\text{span}}\{\psi_{0,k} : k \in \mathbb{Z}\}$$

Let us recall the property $(W)$:

$$\psi(t)=\sum_{k=-\infty}^{\infty}(-1)^k c_{1-k}\phi(2t-k).$$

**Proof 10.2.5:** **Assertion 1:** For all $n,m$ in $\mathbb{Z}$, $\phi_{0,n}\perp\psi_{0,m}$. Indeed,

$$\begin{aligned}
\langle\phi_{0,n},\,\psi_{0,m}\rangle &= \int_{\mathbb{R}}\phi(t-n)\psi(t-m)\mathrm{d}t\\
&= \int_{\mathbb{R}}\left[\sum_k c_k\phi(2t-2n-k)\sum_\ell(-1)^\ell c_{1-\ell}\phi(2t-2m-\ell)\right]\mathrm{d}t\\
&= \sum_k\sum_\ell(=-1)^\ell c_k c_{1-\ell}\int_{\mathbb{R}}\phi(x-2n-k)\phi(x-2m-\ell)\frac{1}{2}\mathrm{d}x\\
&\quad(\text{Put } 2t=x)\\
&= \frac{1}{2}\sum_k\sum_\ell(-1)^\ell c_k c_{1-\ell}\delta_{2n+k,2m+\ell}\\
&= \frac{1}{2}\sum_k(-1)^k c_k c_{p-k}\\
&\quad(2n+k=2m+\ell\Rightarrow\ell=2n-2m+k.\ \text{Put } p=2m-2n+1.)\\
&= \frac{1}{4}\left[\sum_k(-1)^k c_k c_{p-k}+\sum_i(-1)^i c_i c_{p-i}\right]\quad(\text{Put } p-i=q.)\\
&= \frac{1}{4}\left[\sum_k(-1)^k c_k c_{p-k}+\sum_q(-1)^{p-q} c_q c_{p-q}\right]\\
&= \frac{1}{4}\left[\sum_k(-1)^k c_k c_{p-k}-\sum_q(-1)^{q} c_q c_{p-q}\right]=0.
\end{aligned}$$

We have just established that $V_0\perp W_0$.

**Assertion 2:** $\{\psi_{0,n}:n\in\mathbb{Z}\}$ is orthonormal. We have

$$\begin{aligned}
\phi_{0,n}(t)=\phi(t-n) &= \sum_k c_k\phi(2t-2n-k)=2^{-1/2}\sum_k c_k\phi_{1,k+2n},\\
\psi_{0,n}(t)=\psi(t-n) &= \sum_\ell(-1)^\ell c_{1-\ell}\phi(2t-2n-\ell)=2^{-1/2}\sum_\ell(-1)^\ell c_{1-\ell}\phi_{1,\ell+2n}.
\end{aligned}$$

Hence

$$\begin{aligned}
\langle\phi_{0,n},\,\phi_{0,m}\rangle &= 2^{-1}\langle\sum_k c_k\phi_{1,k+2n},\,\sum_\ell c_\ell\phi_{1,\ell+2m}\rangle\\
&= 2^{-1}\sum_k\sum_\ell c_k c_\ell\delta_{k+2n,\ell+2m}=2^{-1}\sum_k c_k c_{k+2n-2m}.
\end{aligned}$$

A similar calculation gives

$$\begin{aligned}\langle \psi_{0,n}, \psi_{0,m} \rangle &= 2^{-1} \sum_k (-1)^k (-1)^{k+2n-2m} c_{1-k} c_{1-2n+2m-k} \\ &= 2^{-1} \sum_i C_i C_{i+2m-2n} = \delta_{m,n}.\end{aligned}$$

**Assertion 3:**

$$V_1 = V_0 \overset{\perp}{\bigoplus} W_0.$$

From $(S_1)$ and $(W)$ we have $\phi \in V_1, \psi \in V_1$. The same is true for their integer shifts. Hence $V_0 + W_0 \subset V_1$. By Lemma 10.2.4, $V_0 + W_0$ is closed. Also, by Lemma 10.2.3, $\phi_{1,k} \in V_0 + W_0$. This implies $V_1 = \overline{\text{span}}\{\phi_{1,k}\} \subset V_0 + W_0$. Since $V_0 \perp W_0$, we conclude that $V_0 \bigoplus^{\perp} W_0 = V_1$. To complete the proof, let $W_j = \overline{\text{span}}\{\psi_{j,k} : k \in \mathbb{Z}\}$. On similar lines as before, one obtains

$$V_j = V_{j-1} \overset{\perp}{\bigoplus} W_{j-1}, \quad j \in \mathbb{Z}.$$

**Assertion 4:** $\langle \psi_{j,k}, \psi_{i,\ell} \rangle = \delta_{j,i} \delta_{k,\ell}$.

For $j = i$, this follows from Lemma 10.1.1. Assume $j \neq i$. Let $j < i$. Then $\psi_{j,k} \in W_j \subset V_{j+1} \subset V_i$ and $\psi_{i,\ell} \in W_i$. Hence $\psi_{j,k} \perp \psi_{i,\ell}$.

**Remark 10.2.1:** Assume $\phi \in L^2(\mathbb{R})$ satisfies properties $(S_0)$ and $(S_1)$. Then for $\ell \in \mathbb{Z}$,

$$\begin{aligned}\phi_{0,\ell}(t) &= \phi(t-\ell) = \sum_k c_k \phi(2t - 2\ell - k) \\ &= 2^{-1/2} \sum_k c_k 2^{1/2} \phi(2t - 2\ell - k) = 2^{-1/2} \sum_k c_k \phi_{1,2\ell+k}. \\ \phi_{j-1,k}(t) &= 2^{\frac{j-1}{2}} \phi(2^{j-1} t - k) \\ &= 2^{\frac{j-1}{2}} \sum_\ell c_\ell \phi(2^j t - 2k - \ell) \\ &= 2^{-1/2} \sum_\ell c_\ell \phi_{j,2k+\ell}.\end{aligned}$$

Thus

$$\phi_{j-1,k}(t) = 2^{-1/2} \sum_r c_{r-2k} \phi_{j,r}. \tag{10.2.4}$$

On the same lines, one sees that

$$\psi_{j-1,k} = 2^{-1/2} \sum_r (-1)^r c_{2k+1-r} \phi_{j,r}. \tag{10.2.5}$$

**Theorem 10.2.2.** *Let $\phi$ in $L^2(\mathbb{R})$ satisfy properties $(S_0)$ and $(S_1)$. Let $\psi$ in $L^2(\mathbb{R})$ be defined by*

$$(W) \qquad \psi(t) = \sum_{k=-\infty}^{\infty} (-1)^k c_{1-k} \phi(2t-k)$$

*(with coefficients $c_j$'s as in $(S_1)$).*

*In addition, assume that*

$$\bigcap_{j\in\mathbb{Z}} V_j = \{0\} \text{ and } \overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R}).$$

*Then $\{\psi_{j,k} : j,k \in \mathbb{Z}\}$ is an o.n. wavelet for $L^2(\mathbb{R})$.*

**Proof 10.2.6:** In view of the previous theorem, we need only prove that the orthonormal set $\{\psi_{j,k} : j,k \in \mathbb{Z}\}$ is *complete*. To this end, we need only show that

$$f \in L^2(\mathbb{R}), f \perp \psi_{j,k}, j,k \in \mathbb{Z} \Rightarrow f = 0.$$

Assuming these hypotheses, we have $f \perp W_j$, for every $j \in \mathbb{Z}$. For each $j \in \mathbb{Z}$, let $P_j : L^2(\mathbb{R}) \to V_j$ denote the orthogonal projector onto $V_j$, and let $v_j := P_j f$. Thus $v_j \in V_j$ and $f - v_j \perp V_j$. Since

$$V_j = V_{j-1} \overset{\perp}{\bigoplus} W_{j-1},$$

we have

$$f - v_j \perp V_{j-1} \text{and} f - v_j \perp W_{j-1} \Rightarrow v_j \in W_{j-1}^{\perp}.$$

Since $f \perp W_{j-1}$, we have $v_j \in V_{j-1}$. Also,

$$v_j \in V_{j-1} \text{ and } f - v_j \perp V_{j-1} \Rightarrow v_j = v_{j-1}.$$

Thus $\{v_j\}$ is a constant sequence. But the density of $\bigcup V_j$ and the nested property $V_j \subset V_{j+1}$, for all $j \in \mathbb{Z}$ of $V_j$'s entail $v_j \to f$. Hence $v_j = f$ for all $j \in \mathbb{Z}$, from which one concludes that $f \in \bigcap_j V_j$. Thus $f = 0$.

**Example 10.2.1.** Perhaps, one simplest pair of functions illustrating the previous theorem is

$$\psi = \psi^H (\textit{the Haar wavelet}), \;\; \phi = \chi_{[0,1)}.$$

It is easy to verify that $\phi$ obeys the simple two-scale relation

$$\phi(t) = \phi(2t) + \phi(2t-1).$$

Thus here, $c_0 = c_1 = 1$ and $c_k = 0$ for every $k \in \mathbb{Z} \setminus \{0,1\}$, and $\psi$ is given by

$$\psi(t) = \phi(2t) - \phi(2t-1),$$

which is none other than the Haar wavelet.

It remains to check in the above example, the two properties

$$\bigcap_j V_j = \{0\} \text{ and } \overline{\bigcup V_j} = L^2(\mathbb{R}).$$

Here

$$\phi_{0,k} = \begin{cases} 1, \ k \le t < k+1 \\ 0, \text{ otherwise.} \end{cases}$$

Thus

$$V_0 := \left\{ f \in L^2(\mathbb{R}) : f \text{ constant on } [k, k+1), \forall k \in \mathbb{Z} \right\}$$

and

$$V_j := \left\{ f \in L^2(\mathbb{R}) : f \text{ constant on } [\frac{k}{2^j}, \frac{k+1}{2^j}), \forall k \in \mathbb{Z} \right\}.$$

Clearly,

$$\begin{aligned} f \in V_0 &\Rightarrow f \text{ constant on } [k, k+1), \forall k \in \mathbb{Z} \\ &\Rightarrow f \text{ constant on } [\frac{k}{2}, \frac{k+1}{2}), \forall k \in \mathbb{Z} \\ &\Rightarrow f \ \in V_1. \end{aligned}$$

Thus $V_0 \subset V_1$. Likewise, $V_j \subset V_{j+1}, \forall j \in \mathbb{Z}$. Clearly, the space $\mathcal{S}$ of *step functions* is dense in $\overline{\bigcup_j V_j}$. It is well known that $\mathcal{S}$ is also dense in $L^2(\mathbb{R})$. Hence

$$\overline{\bigcup V_j} = L^2(\mathbb{R}).$$

Moreover,

$$f \in \bigcap_j V_j \Rightarrow f = \text{constant on } [0, \frac{1}{2^j}), \forall j \in \mathbb{Z}.$$

Letting $j \to -\infty$, we get $f =$ constant on $[0, \infty)$. Since $f \in L^2(\mathbb{R})$, this constant must be zero. It follows by a similar argument that $f$ is identically 0 on $(-\infty, 0\}$. Thus,

$$\bigcap_j V_j = \{0\}.$$

**Remark 10.2.2:** We note that here it is easy to check directly that $\phi$ satisfies $(S_0)$: the set $\{\phi_{0,k} : k \in \mathbb{Z}\}$ is orthonormal.

## 10.3 Classification of Wavelets and Multiresolution Analysis

Let us recall that if $X$ is a separable Hilbert space, then a (Schauder) basis $\{x_n\}$ of $X$ is said to be a **Riesz basis** of $X$ if it is *equivalent* to an orthonormal basis $\{u_n\}$ of $X$, in the sense that, there exists a bounded invertible operator $T : X \to X$ such that $T(x_n) = u_n, \forall n \in \mathbb{N}$. From this definition, it is easy to prove the next proposition.

**Proposition 10.3.1** Let $X$ be a separable Hilbert space. Then the following statements are equivalent.

(a) $\{x_n\}$ is a Riesz basis for $X$.

(b) $\overline{\text{span}}\{x_n\} = X$ and for every $N \in \mathbb{N}$ and arbitrary constants $c_1, c_2, \ldots, c_N$, there are constants $A, B$ with $0 < A \le B < \infty$ such that

$$A \sum_{i=1}^{N} |c_i|^2 \le \left\| \sum_{i=1}^{N} c_i x_i \right\|^2 \le B \sum_{i=1}^{N} |c_i|^2 .$$

**Remark 10.3.1:** Let $\{x_n\}$ be a Riesz basis in $X$. Then the series $\sum_{i=1}^{\infty} c_i x_i$ is convergent in $X$ if and only if $c = \{c_i\} \in \ell^2$. As a result, each $x \in X$ has a unique representation

$$x = \sum_{i=1}^{\infty} c_i x_i, \;\; c = \{c_i\} \in \ell^2.$$

The preceding discussion enables one to define a *Riesz basis* of $X$ as follows.

**Definition 10.3.1.** A sequence $\{x_n\}$ in a Hilbert space $X$ is said to constitute a **Riesz basis** of $X$ if $\overline{\text{span}}\{x_n\}_{n\in\mathbb{N}} = X$ and there exists constants $A, B$ with $0 < A \le B < \infty$ such that

$$A \sum_{i=1}^{\infty} \left|c_j\right|^2 \le \left\| \sum_{j=1}^{\infty} c_j x_j \right\|^2 \le B \sum_{j=1}^{\infty} \left|c_j\right|^2$$

for every sequence $c = \{c_j\} \in \ell^2$.

We are now ready for the following definition.

**Definition 10.3.2.** A function $\psi$ in $L^2(\mathbb{R})$ is called an $\mathcal{R}$**-function** if $\{\psi_{j,k} : j,k \in \mathbb{Z}\}$ as defined in (10.1.3) is a Riesz basis of $L^2(\mathbb{R})$ in the sense that

$$\overline{\text{span}}\{\psi_{j,k} : j,k \in \mathbb{Z}\} = L^2(\mathbb{R}),$$

and

$$A\|\{c_{j,k}\}\|^2_{\ell^2(\mathbb{Z})} \le \| \sum_{j,k\in\mathbb{Z}} c_{j,k}\psi_{j,k}\|^2 \le B\|\{c_{j,k}\}\|^2_{\ell^2(\mathbb{Z})}$$

holds for all doubly bi-infinite sequences $\{c_{j,k}\} \in \ell^2(\mathbb{Z})$ and for suitable constants $A, B$ such that $0 < A \le B < \infty$.

Next suppose that $\psi$ is an $\mathcal{R}$-function. By Hahn-Banach theorem, one can show that there exists a unique Riesz basis $\{\psi^{j,k}\}$ of $L^2(\mathbb{R})$ which is dual to the Riesz basis $\{\psi_{j,k}\}$ :

$$\langle \psi_{j,k}, \, \psi^{l,m} \rangle = \delta_{j,l}\delta_{k,m}, \; j,k,l,m \in \mathbb{Z}. \tag{10.3.1}$$

Consequently, every function $f$ in $L^2(\mathbb{R})$ admits the following(unique) series representation:

$$f(t) = \sum_{j,k=-\infty}^{\infty} \langle f, \psi_{j,k} \rangle \psi^{j,k}. \tag{10.3.2}$$

Note that, although the coefficients in this expansion are the values of CWT of $f$ relative to $\psi$, the series (10.3.2) is, in general, *not a wavelet series*. In order that this be a wavelet series, there must exist some function $\tilde{\psi}$ in $L^2(\mathbb{R})$ such that

$$\psi^{j,k} = \tilde{\psi}_{j,k}, \quad j,k \in \mathbb{Z},$$

where $\tilde{\psi}_{j,k}$ is as defined in (10.1.3) from the function $\tilde{\psi}$.

Clearly, if $\{\psi_{j,k}\}$ is an o.n. basis of $L^2(\mathbb{R})$, then (10.3.1) holds with $\psi^{j,k} = \psi_{j,k}$, or $\tilde{\psi} = \psi$. In general, however, such a $\tilde{\psi}$ *does not exit.*

If $\psi$ is chosen such that $\tilde{\psi}$ exists, then the pair $(\psi, \tilde{\psi})$ gives rise to the following convenient (dual) representation:

$$\begin{aligned} f(t) &= \sum_{j,k=-\infty}^{\infty} \langle f, \psi_{j,k} \rangle \tilde{\psi}_{j,k} \\ &= \sum_{j,k=-\infty}^{\infty} \langle f, \tilde{\psi}_{j,k} \rangle \psi_{j,k} \end{aligned}$$

for any element $f$ of $L^2(\mathbb{R})$.

**Definition 10.3.3.** A function $\psi$ in $L^2(\mathbb{R})$ is called an **$\mathcal{R}$-wavelet** (or simply a **wavelet**) if it is an $\mathcal{R}$-function and there exists a function $\tilde{\psi}$ in $L^2(\mathbb{R})$, such that $\{\psi_{j,k}\}$ and $\{\tilde{\psi}_{j,k}\}$, as defined in (10.1.3), are dual bases of $L^2(\mathbb{R})$. If $\psi$ is an $\mathcal{R}$-wavelet, then $\tilde{\psi}$ is called a **dual wavelet** corresponding to $\psi$.

**Remark 10.3.2:** A dual wavelet $\tilde{\psi}$ is unique and is itself an $\mathcal{R}$-wavelet. Moreover, $\psi$ is the dual wavelet of $\tilde{\psi}$.

**Remark 10.3.3:** Every wavelet $\psi$, orthonormal or not, generates a "wavelet series" expansion of any $f$ in $L^2(\mathbb{R})$:

$$f(t) = \sum_{j,k=-\infty}^{\infty} c_{j,k} \psi_{j,k}(t), \tag{10.3.3}$$

where each $c_{j,k}$ is the CWT of $f$ relative to the dual $\tilde{\psi}$ of $\psi$ evaluated at $(a,b) = \left(\frac{1}{2^j}, \frac{k}{2^j}\right)$.

We are now ready to look at an important decomposition of the space $L^2(\mathbb{R})$. Let $\psi$ be any wavelet and consider the Riesz basis $\{\psi_{j,k}\}$ that it generates. For each $j \in \mathbb{Z}$, let

$$W_j = \overline{\text{span}}\{\psi_{j,k} : k \in \mathbb{Z}\}.$$

(10.3.3) suggests that $L^2(\mathbb{R})$ can be decomposed as a *direct sum* of the spaces $W_j$'s:

$$L^2(\mathbb{R}) = \bigoplus_{j\in\mathbb{Z}} W_j \tag{10.3.4}$$

in the sense that every $f$ in $L^2(\mathbb{R})$ has a unique decomposition

$$f(t) = \ldots + g_{-1} + g_0 + g_1 + \ldots$$

where $g_j \in W_j, \forall j \in \mathbb{Z}$.

Moreover, if $\psi$ is an o.n. wavelet, then in the above decomposition, the direct sum is, in fact, an *orthogonal direct sum*:

$$L^2(\mathbb{R}) = \overset{\perp}{\bigoplus_{j\in\mathbb{Z}}} W_j := \ldots \overset{\perp}{\bigoplus} W_{-1} \overset{\perp}{\bigoplus} W_0 \overset{\perp}{\bigoplus} W_1 \ldots. \tag{10.3.5}$$

Note that here, for $\forall j, l \in \mathbb{Z}, j \neq l$,

$$W_j \cap W_l = \{0\}, \;\; W_j \perp W_l.$$

**Definition 10.3.4.** A wavelet $\psi$ in $L^2(\mathbb{R})$ is called a **semi-orthogonal wavelet** (or **s.o. wavelet**) if the Riesz basis $\{\psi_{j,k}\}$ that it generates satisfies

$$\langle \psi_{j,k}, \psi_{l,m} \rangle = 0, \;\; j \neq l \;\; j,k,l,m \in \mathbb{Z} \tag{10.3.6}$$

Clearly, a semi-orthogonal wavelet also gives rise to an *orthogonal decomposition* (10.3.5) of $L^2(\mathbb{R})$.

We now come to the important concept of *multiresolution analysis* first introduced by Meyer(1986) and Mallat(1989). We saw that any wavelet $\psi$ (semiorthogonal or not) generates a direct sum decomposition (10.3.4) of $L^2(\mathbb{R})$.

For each $j \in \mathbb{Z}$, let us consider the closed subspaces

$$V_j = \ldots \bigoplus W_{j-2} \bigoplus W_{j-1}$$

of $L^2(\mathbb{R})$. These subspaces satisfy the following properties:

(MR1) $\ldots \subset V_{-1} \subset V_0 \subset V_1 \subset \ldots$;
(MR2) $\overline{\cup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R})$, the closure being taken in the topology of $L^2(\mathbb{R})$;
(MR3) $\cap_{j\in\mathbb{Z}} V_j = \{0\}$;
(MR4) $V_{j+1} = V_j \bigoplus W_j, \;\; j \in \mathbb{Z}$; and
(MR5) $f(t) \in V_j \Leftrightarrow f(2t) \in V_{j+1}, \;\; j \in \mathbb{Z}$.

If the initial subspace $V_0$ is generated by a single function $\phi$ in $L^2(\mathbb{R})$ in the sense that

$$V_0 = \overline{\text{span}}\{\phi_{0,k} : k \in \mathbb{Z}\}, \tag{10.3.7}$$

then using (MR5) all the subspaces $V_j$ are also generated by the same $\phi$:

$$V_j = \overline{\text{span}}\{\phi_{j,k} : k \in \mathbb{Z}\}, \text{where } \phi_{j,k}(t) = 2^{\frac{j}{2}} \phi(2^j t - k). \tag{10.3.8}$$

**Definition 10.3.5.** A function $\phi$ in $L^2(\mathbb{R})$ is said to generate a **multiresolution analysis (MRA)** if it generates a ladder of closed subspaces $V_j$ that satisfy (MR1), (MR2), (MR3) and (MR5) in the sense of (10.3.8), and such that the following property holds.

(MR0) $\{\phi_{0,k} : k \in \mathbb{Z}\}$ forms a Riesz basis of $V_0$.

This means, there must exist constants $A, B$, with $0 < A \le B < \infty$ such that

$$A\,\|\{c_k\}\|^2_{\ell^2(\mathbb{Z})} \le \left\| \sum_{k\in\mathbb{Z}} c_k \phi_{0,k} \right\|_2^2 \le B\,\|\{c_k\}\|^2_{\ell^2(\mathbb{Z})} \tag{10.3.9}$$

for all bi-infinite sequences $c = \{c_k\} \in \ell^2(\mathbb{Z})$.

In this case, $\phi$ is called a **scaling function**.

Using the Poisson's lemma (cf., e.g., [2], Lemma 2,24) and the Parseval's identity for Fourier transforms one shows that for any $\phi$ in $L^2(\mathbb{R})$, the following hold:

(A) The set $\{\phi(x-k) : k \in \mathbb{Z}\}$ is orthonormal.
$\Leftrightarrow$
The Fourier transform $\hat{\phi}$ of $\phi$ satisfies the identity

$$\sum_{-\infty}^{\infty} |\hat{\phi}(\omega + 2\pi k)|^2 = 1, \tag{10.3.10}$$

for almost all $\omega \in \mathbb{R}$.

(B) The family of functions $\{\phi(x-k) : k \in \mathbb{Z}\}$ satisfies the Riesz condition (10.3.9) with Riesz bounds $A$ and $B$.
$\Leftrightarrow$
The Fourier transform $\hat{\phi}$ of $\phi$ satisfies

$$A \le \sum_{-\infty}^{\infty} |\hat{\phi}(\omega + 2\pi k)|^2 \le B, \text{a.e.} \tag{10.3.11}$$

**Remark 10.3.4:** The condition (MR5) implies

$$f(t) \in V_0 \Leftrightarrow f(2^j t) \in V_j.$$

**Remark 10.3.5:** The spaces $V_j$ possess the following shift invariance property:

$$f(t) \in V_j \Leftrightarrow f(t + \frac{k}{2^j}) \in V_j, \forall k \in \mathbb{Z}.$$

The above remark follows from:

$$\begin{aligned} \phi_{j,\ell}(t + \frac{k}{2^j}) &= 2^{\frac{j}{2}} \phi(2^j \left(t + \frac{k}{2^j}\right) - \ell) \\ &= 2^{\frac{j}{2}} \phi(2^j t - (\ell - k)). \end{aligned}$$

Next, we give a few examples of MRA of $L^2(\mathbb{R})$.

For $j, k \in \mathbb{Z}$, let us denote by $I_{j,k}$ the interval $[\frac{k}{2^j}, \frac{k+1}{2^j})$.

**Example 10.3.1.** For each $j \in \mathbb{Z}$, let $V_j$ denote the space of piecewise constants:

$$V_j = \{f \in L^2(\mathbb{R}) : f|_{I_{j,k}} \equiv \text{ constant}, \forall k \in \mathbb{Z}\}.$$

Here $V_0$ is the closed linear span of the integer shifts of the characteristic function $\chi_{[0,1]}$, which is the scaling function $\phi$. Here, it is easily verified that the set $\{\phi_{0,k} : k \in \mathbb{Z}\}$ is orthonormal, and we have already checked that $\{V_j : j \in \mathbb{Z}\}$ is a multiresolution. In this case, the wavelet is the *Haar wavelet*, which is, in fact, an o.n. wavelet.

**Example 10.3.2.** For each $j \in \mathbb{Z}$, let $V_j$ be the $L^2(\mathbb{R})$-closure of the set $S_j$:

$$S_j = \{f \in L^2(\mathbb{R}) \cap \mathcal{C}(\mathbb{R}) : f|_{I_{j,k}} \text{is linear}, \forall k \in \mathbb{Z}\}.$$

It is easy to check all the conditions of MRA except (MR0) similar to the previous example. Checking of (MR0) involves computation of Riesz bounds, which in this case, are $A = \frac{1}{3}, B = 1$. Here the scaling function $\phi$ can be taken to be the hat function:

$$\phi(t) = \begin{cases} t, \text{ if } 0 \leq t \leq 1 \\ 2-t, \text{ if } 1 \leq t \leq 2 \\ 0, \text{ otherwise.} \end{cases}$$

Note that here

$$\phi_{o,k}(t) = \phi(t-k) = \begin{cases} t-k, \ k \leq t \leq k+1 \\ k+2-t, \ k+1 \leq t \leq k+2 \\ 0, \text{ otherwise.} \end{cases}$$

It is easy to see that the set $\{\phi_{0,k} : k \in \mathbb{Z}\}$ is not orthonormal. Here using the two-scale relation and a variant of Theorem 10.2.2, we can show that the corresponding wavelet $\psi$ is given by

$$\psi(t) = \phi(2t) - \frac{1}{2}\phi(2t-1) - \frac{1}{2}\phi(2t+1)$$

whose support is $[-\frac{1}{2}, \frac{3}{2}]$.

## 10.4 Spline Wavelets

For each positive integer $m$, we denote by $\mathcal{S}_m(2^{-j}\mathbb{Z}) =: \mathcal{S}_m^j$ the space of **cardinal splines** of order $m$ and with the knot sequence $2^{-j}\mathbb{Z}$, for a fixed $j \in \mathbb{Z}$ :

$$\mathcal{S}_m(2^{-j}\mathbb{Z}) = \{f \in \mathcal{C}^{m-2}(\mathbb{R}) : f|_{I_{j,k}} \in \mathcal{P}_m, \forall k \in \mathbb{Z}\}$$

(Here $\mathcal{P}_m$ denotes the class of polynomials of *order m , i.e.* of degree $\leq m-1$.) For each $m \in \mathbb{N}$, the $m^{\text{th}}$ **order cardinal B-spline** $N_m$ is defined by

$$N_m = \chi_{[0,1)} * \ldots * \chi_{[0,1)} \text{ ( } m \text{ times convoluted).}$$

Put differently, $N_m$ is defined recursively by:

$$N_m(t) = \int_{-\infty}^{\infty} N_{m-1}(t-s)N_1(s)\mathrm{d}s$$
$$= \int_0^1 N_{m-1}(t-s)\mathrm{d}s, \text{with } N_1 := \chi_{[0,1)}.$$

The scaling functions in the two examples in the previous section are respectively the first order and the second order cardinal $B$-spline. It is well known that any $f \in \mathcal{S}_m^j$ can be written as

$$f(t) = \sum_k c_k N_m(t-k). \tag{10.4.1}$$

Taking $N_m$ as the scaling function, let us define

$$V_0^m = \overline{\text{span}}\mathcal{S}_m^0 = \mathcal{S}_m(\mathbb{Z}) \tag{10.4.2}$$

Hence, a function $f$ is in $V_0^m$ if and only if it has a $B$-spline series representation (10.4.1) with the coefficient sequence $c = \{c_k\} \in \ell^2(\mathbb{Z})$. The other spaces $V_j^m$ are defined by

$$f(t) \in V_j^m \Leftrightarrow f(2t) \in V_{j+1}^m, \quad j \in \mathbb{Z}.$$

In other words,

$$V_j^m = \overline{\text{span}}\mathcal{S}_m^j.$$

Clearly the subspaces $\{V_j^m : j \in \mathbb{Z}\}$ satisfy (MR1). The verification of (MR2) is immediate: The class of polynomials $\mathcal{P}$ is dense in $L^2(\mathbb{R})$ and $\mathcal{P} \subset V_j^m, \forall j \in \mathbb{Z}$. This implies

$$\overline{\bigcup_{j\in\mathbb{Z}} V_j^m} = L^2(\mathbb{R}).$$

The verification of (MR3) is exactly as in Example 10.3.1 The verification of (MR0) is carried out as in Example 10.3.2 with $\phi$ replaced by $N_m$. Here the smallest value of $B$ is 1, and the largest value of A can be expressed in terms of the roots of the *Euler-Frobenius polynomial*

$$E_{2m-1}(z) = (2m-1)!z^{m-1} \sum_{k=-m+1}^{m-1} N_{2m}(m+k)z^k \tag{10.4.3}$$

From the nested sequence of spline spaces $V_j^m$, we have the orthogonal complementary subspaces $W_j^m$, given by

$$W_{j+1}^m = V_j^m \bigoplus W_j^m, \quad j \in \mathbb{Z}.$$

Just as the $B$-spline $N_m$ is the minimally supported generator of $\{V_j^m\}$, we are interested in finding the minimally supported $\psi_m \in W_0$ that generates the mutually orthogonal subspaces $W_j$. Such compactly supported functions $\psi_m$ are called the ***B*-wavelets** of order $m$. It turns out that

$$\text{support } \mathrm{N_m} = [0,\mathrm{m}]; \quad \text{support } \psi_\mathrm{m} = [0, 2\mathrm{m}-1], \forall \mathrm{m} \in \mathbf{N}$$

We mention without working out further details that

$$\psi_m(t) = \sum_{k=0}^{3m-2} q_k N_m(2t-k), \tag{10.4.4}$$

with

$$q_k = q_k^{(m)} = \frac{(-1)^k}{2^{m-1}} \sum_{l=0}^{m} \binom{m}{l} N_{2m}(k-l+1). \tag{10.4.5}$$

For the relevant details, we refer the reader to Chui (1992, Chapter 6).

## 10.5 A Variant of Construction of Orthonormal Wavelets

Let us go back once again to Theorems 10.2.1 and 10.2.2. Suppose $\phi$ in $L^2(\mathbb{R})$ is such that $\phi$ does not satisfy $(S_0)$, i.e., $\{\phi_{0,k} : k \in \mathbb{Z}\}$ is *not orthonormal*.

In this case, it seems natural to define $\Phi$ by requiring its Fourier transform to be using (10.3.10) and the Plencherel's thorem,

$$\hat{\Phi}(\omega) = \frac{\hat{\phi}(\omega)}{\{\sum_k |\hat{\phi}(\omega+2\pi k)|^2\}^{1/2}}, \quad \omega \in \mathbb{R}. \tag{10.5.1}$$

**Theorem 10.5.1.** *Let $\phi$ in $L^2(\mathbb{R})$ be such that it satisfies*

*(MR0):* $\{\phi_{0,k} : k \in \mathbb{Z}\}$ *is a Riesz basis of* $V_0$.

*Define $\Phi \in L^2(\mathbb{R})$ by (10.5.1). Then $\{\Phi_{0,k} : k \in \mathbb{Z}\}$ is an orthonormal basis for the space $V_0$.*

**Theorem 10.5.2.** *Let $\phi$ in $L^2(\mathbb{R})$ be such that $\{\phi_{0,k} : k \in \mathbb{Z}\}$ is a Riesz basis for $V_0$ and suppose $\phi \in V_1$. Then $\Phi$ as defined in (10.5.1) satisfies a two-scale relation*

$$\Phi(t) = \sum_{k=-\infty}^{\infty} a_k \Phi(2t-k), \quad (a = \{a_k\} \in \ell^2(\mathbb{Z})).$$

*Let $\Psi$ be defined by*

$$\Psi(t) = \sum_k (-1)^k a_{1-k} \Phi(2t-k). \tag{10.5.2}$$

*Then the set $\{\Psi_{j,k} : j,k \in \mathbb{Z}\}$ is orthonormal.*

*Furthermore, if $V_j$'s satisfy (MR2) and (MR3), then $\Psi$ is an orthonormal wavelet.*

The proofs of the above theorems follow from Theorems 10.2.1 and 10.2.2 by applying (10.3.10) and (10.3.11). The details are left to the reader as exercises.

# Exercises 10.5

**10.5.1.** Let $\psi^H$ be the Haar wavelet. Show that for integers $n < m$,

$$\int_n^m \psi^H(t)\mathrm{d}t = 0.$$

**10.5.2.** Let $f, g \in L^2(\mathbb{R})$, and suppose that $f_{oj} \perp g_{0i}, \forall i, j \in \mathbb{Z}$. Show that $f_{nj} \perp g_{ni}, \forall n, i, j \in \mathbb{Z}$.

**10.5.3.** Let $\psi \in L^2(\mathbb{R}), n \in \mathbb{Z}, i \in \mathbb{Z}$. Define $\phi = \psi_{ni}$. Show that

$$\mathrm{span}\{\phi_{\mathrm{kj}} : \mathrm{k,j} \in \mathbb{Z}\} = \mathrm{span}\{\psi_{\mathrm{kj}} : \mathrm{k,j} \in \mathbb{Z}\}.$$

**10.5.4.** Let $\{u_n\}$ be an orthonormal sequence in a Hilbert space. Let $\alpha, \beta$ be in $\ell^2$ such that $\alpha \perp \beta$. Define

$$w = \sum_k \alpha_k u_k, \quad v = \sum_k \beta_k u_k.$$

Show that $w \perp v$. Is the converse true?

**10.5.5.** Verify directly the orthonormality of the family of functions $\{\psi^H_{j,k}; j, k \in \mathbb{Z}\}$.

**10.5.6.** Give a proof of Theorem 10.5.1.

**10.5.7.** Give a proof of Theorem 10.5.2.

## References

Calderòn, A. (1964). Intermediate spaces and interpolation, the complex method, *Studia Math.* **24**, 113–190.

Chui, C.K. (1992). *"An introduction to wavelets"*, Academic Press, San Diego.

Chui, C.K. and Wang, J.Z. (1993) An analysis of cardinal-spline wavelets, *J. Approx. Theory,* **72(1)**, 54–68.

Daubechies, I. (1992). *"Ten lectures on wavelets", SIAM*, Philadelphia.

Daubechies, I. (1998). Orthonormal bases of compactly supported wavelets, *Comm. Pure Appl.Math.* **41**, 909–996.

Mallat, S. (1998). *"A wavelet tour of signal processing",* Academic Press, New York.

Mallat, S. (1989). Multiresolution approximations and wavelet orthonormal bases of $L^2(\mathbb{R})$, *Trans. Amer. Math. Soc.* **315**, 69–87.

Meyer, Y. (1992). *"Wavelets and Operators"* Cambridge University Press.

Grossman, A. and Morlet, J. (1984). Decomposition of Hardy functions into square integrable wavelets of constant shape, *SIAM J. Math. Anal.* **15**, 723–736.